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Explanatory Strategies of Preservice Mathematics Teachers about Divisibility by Zero

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Abstract

In this study, it was aimed to reveal the explanatory strategies that preservice teachers use in the process of explaining the concept of divisibility by zero. It was investigated how the concept of divisibility by zero, which can be used in expressing the case where the denominator is present in the definition of important concepts of the secondary school curriculum such as the fraction and rational number, is defined and explained. A scale consisting of three open-ended questions, in which it was questioned what the definition of the concept of divisibility by zero is and how this concept can be explained to the secondary school/high school student, was used as a data collection instrument. The data were collected through this scale and the content analysis method was adopted in the data analysis. As a result of the analyses made, it was determined that the preservice teachers use the rule strategy the most on the subject of divisibility by zero.

Keywords: zero number; strategies; preservice teachers.

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INTRODUCTION

One of the most important factors determining the quality of teaching is the knowledge of the teacher. Shulman (1986) expressed that the knowledge that teachers should have is the field knowledge, knowledge of understanding the student, and field-specific pedagogical knowledge. While the field knowledge is the teacher's knowledge of understanding the mathematics with its concepts, principles, and rules (Ball, 1990; Ma, 1999), the pedagogical field knowledge is the teacher's knowledge of how to teach using also the field knowledge. Teacher's explaining the mathematical concepts with their most appropriate forms of representation to students and being able to give the most powerful examples and make the most powerful explanations in explaining the mathematical concepts depends on the teacher's field knowledge. Therefore, the pedagogical content knowledge requires the field knowledge. However, that does not mean that the pedagogical knowledge of an individual with good field knowledge is also good. In addition to good mathematics knowledge, the pedagogical knowledge in which a teacher realizes teaching with explanations that are appropriate for the student level (Baki, 2013), will be significant. The pedagogical field knowledge includes all the educational activities, skills, and features that the teacher has, such as the ability to transfer knowledge effectively to students, that is, to turn the knowledge into a form that the student can easily understand (McDiarmid, Ball, and Anderson, 1989). One of the most important components of the pedagogical field knowledge is making appropriate educational explanations associated with mathematical concepts, principles, theorems, and rules. Much of the research in the literature has shown that explanations that teachers and preservice teachers use in teaching are based on rote-learning rather than conceptual understanding, that is, they are more rule and operation-based (Kinach, 2002). Educational explanations related to the concepts teachers use in the process of mathematics teaching are important because knowledge of the mathematical knowledge of these individuals can be had by examining the educational explanations that teachers have made on the mathematical concepts. In this study, educational explanations of preservice teachers who will be the teachers of the future were examined. Explanations made by preservice teachers will reveal how they associate what they learned in their abstract algebra lesson with school mathematics and thus, the effects of the lessons they took. Furthermore, these educational explanations of preservice teachers will inform us about how they will transfer this subject to their students when they become teachers in the future.

In their studies, many researchers have revealed that the number zero and teaching this number is quite difficult, teachers and students have problems in interpreting the number zero (Ma, 1999; Quinn, Lamberg, and Perrin, 2008), and that teachers and students have insufficient knowledge of what the division of any number by *0* means (Arsham, 2008; Tsamir, Sheffer, and Tirosh, 2000). Ball (1990) showed that understandings of preservice teachers are based on rote learning rather than conceptual understanding. Although the division by zero is not included in the curriculum as a direct achievement, it is a situation that is encountered by students in the process of defining the rational number, which is among important learning domains of the secondary school and high school. In secondary school textbooks, the rational number is defined as 'a number that can be written as a/b, provided that$a$ and b each being an integer $(b \neq 0)$' (Keskin, 2016). The a/b expression has raised the subject of divisibility. The divisibility in abstract algebra textbooks is defined as '$a,b \in Z$, and if there is $c \in Z$ provided that $a=bc$, then b divides$a$ and it is shown as $b|a$' (Arıkan and Halıcıoğlu, 2012). While there is no problem for students in the case where the denominator is not zero in the divisibility expression, in the case where the denominator is zero, the operation leads us to the undefinability, which seems to be a difficult situation to understand for many students (Tsamir, Sheffer, and Tirosh, 2000). This is because most students at the secondary school level think that the result of all mathematical operations must be a numerical value, and even though individuals at the high school and advanced levels know that it is impossible to divide a number by zero, they have difficulty in explaining it and they tend to explain the situation with answers such as "My teacher said so" (Reys and Grouws, 1975). The division of any number other than zero by 0 is undefined. This can be explained in different ways. For example, if $a/0$ were defined provided that $a \neq 0$,there would be $c \in$ □ with $a/0 = c$. In this case, it would be $a=0.c$, that is, $a=0$. This conflicts with $a \neq 0$. Therefore, the division of a number other than zero by zero is undefined (Kadıoğlu and Kamali, 2009; Özmantar and Bozkurt, 2013). In other words, a division as a/b expresses a multiplication as $a(1/b)$ provided that

$a, b \in$. We know that zero has no inverse in multiplication. There is no real number to multiply with zero and to get 1. In this case, the division such as *a/0* is also undefined (Qiiinn, Lamberg, and Perin, 2008). Therefore, the student who wants to interpret the definition of the rational number must also properly interpret what the division by zero is. Teachers are the ones who teach, organize and shape the learning environment in schools (Zikre and Eu, 2016). This responsibility belongs to the teacher. Such that, incorrect explanations that the teacher uses in the lecturing process can lead to various misconceptions in students (Baştürk and Dönmez, 2011). In this case, first of all, the teacher himself/herself must understand these concepts or processes at the conceptual level (Ma, 1999). In this study, it was aimed to reveal the knowledge that preservice teachers have on the division by zero and which explanatory strategies they use for the concept. Therefore, the problems of the research were determined as follows;

- What are the strategies that preservice teachers use in the meaning of the concept of division by zero and in the process of expressing it to secondary school/high school students?

- What are the abstract mathematical arguments (AMA) strategies that preservice teachers use in the meaning of the concept of division by zero and in the process of expressing it to secondary school/high school students?

- What are the analogy use strategies that preservice teachers use in the meaning of the concept of division by zero and in the process of expressing it to secondary school/high school students?

- What are the rule-based strategies that preservice teachers use in the meaning of the concept of division by zero and in the process of expressing it to secondary school/high school students?

METHOD

Research Model

This study is a qualitative research, and the descriptive review model was adopted as the research design because it was desired to determine the preservice teachers' concept definitions and explanatory strategies related to the divisibility by zero.

Participants of the Research

The participants of the study consisted of 48 individuals who were senior students studying at the department of elementary mathematics teaching and students who graduated from the department of mathematics and attending the pedagogical formation program. All of the participants are individuals who took the abstract algebra lesson and learned the concepts questioned in the study in the class. In the study participants were coded as T1, T2, ..., T48. The first 24 (T1, ..., T24) of these preservice teachers (PT) consisted of those who graduated from the department of mathematics and received the pedagogical formation education and the others (T25, ..., T48) consisted of those who study at the department of mathematics teaching.

Data Collection Instruments and Process

In the study, a form consisting of five open-ended questions that investigate what the definitions of the concepts of prime number and divisibility by zero are and how these concepts can be explained to secondary/high school students was used as a data collection tool and opinions of two different lecturers, one of them being an abstract algebra lecturer, were asked in the creation of the form. Open-ended interview questions including the prime number and divisibility by zero were written on paper and handed out to the participants and preservice mathematics teachers were given one course hour to answer the questions.

Data Analysis

In the study, the content analysis method was used to analyze the data collected in the 2017-2018 fall semester. Cofer's (2015) categories were used for the explanatory strategies that the preservice teachers used in the process of definition and explanation of divisibility by 0. Cofer (2015) expressed the explanatory strategies as the abstract mathematical argument (AMA), analogy, and rules. AMA expresses the use of techniques of abstract mathematical thinking and formal reasoning to explain a definition. In this strategy, mathematical definitions, theorems, axioms, and formulas are used. The analogy is the use of tangible contexts (physical representation) without being intangible to build the reasoning. Rules are the production of individuals with alternative rules and explaining it using expressions such as "It is defined in such a way in the book". Table 1 below shows these strategies and coding examples.

Table 1 Explanatory strategies (Cofer, 2015)

Strategies	Explanations	Example
Abstract Mathematical Argument (AMA)	It is the use of techniques of abstract mathematical thinking and formal reasoning to describe a definition. Mathematical definitions, theorems, axioms, and formulas are used.	1/0=undefined. For example, it equals number a. When a/0=xa=x.0, a≠0, there is no number to give for X, so it is undefined.
Analogy	It is the use of tangible contexts without being intangible to build the reasoning. This context is a physical representation.	…If "1" is divided by "0", it is impossible to get a result since there is no number to divide "1". For example, 5 pieces of candy cannot be divided into non-existent kids, that is, a nullity.
Rules	If individuals produce alternative rules for themselves and explain their rights by using expressions such as "It is defined in such a way in the book".	Number 0 can be divided by all real numbers; however, all real numbers cannot be divided by 0. Their value will be undefined and indefinite. 1/0 has no meaning because it has no known value and it is undefined.

In the data analysis, the first author coded all data according to the specified categories. Thereafter, for the reliability of the coding, half of the data was coded by the second researcher and the inter-rater concordance was calculated to be 94% in the coding using the reliability coefficient calculation formula [Reliability = Consensus/(Consensus+Dissensus)] specified by Miles and Huberman (1994). It can be said that the classification is reliable since the reliability calculations are over 70% (Miles & Huberman, 1994).

FINDINGS

The PTs were addressed questions *"What do you think about divisibility by 0? Interpret the meaning of 1/0."* and *"How do you explain the concept of divisibility by 0 to secondary school or high school students?"*and the answer given by each participant was classified according to the AMA, Analogy, and Rules categories. The answers of the PTs in these categories were explained as the meaning of the division by zero and as the strategies used in the explaining process. The PTs, who interpreted the division by zero, expressed the number in the numerator and denominator as undefined, indefinite, infinite, zero, and no result according to their features. In the first and second questions, 10 participants used the AMA, 11 used analogies, 31 used rules, and 1 left blank (Table 2). In the third question, 1 used the AMA, 14 used analogies, and 20 used rules, 5 used other strategy methods and techniques, and 11 left this question unanswered (Table 2). There was also a shift in the strategies used by some participants in these questions. For example, a student who started with the rule finished the explanation with the AMA or analogy. Furthermore, some participants gave answers that can be included in multiple categories. Therefore, the total number of frequencies was higher than the number of participants.

Table 2 Strategies used in the process of expressing the concept of divisibility by zero

		AMA			Analogies				Rule						O	Empty
		Ud	Id	Ze	Ud	In	NR	O	Ud	Id	In	Ze	Un-In	O		
Zero division means	0/0		2							3						1
	0/number											1				
	1/0	7	1	2	3	7	1		14	1	10		1	1		
	a/0								8	2	1	2				
Total		12			11				44							1
Explanation Process	1/0	1			4	4	1	5	2		1		1		5	11
	a/0								13	1	2					
Total		1			14				20							11

Ud: Undefined, Id: Indefinite, In: Infinite, Ze: Zero, NR: No result, O: Other, Ud-Id: Undefined (Indefinite)

The PTs used the rules the most and AMA strategies the least for division by zero. Some of the preservice teachers addressed the division by zero as separate situations where both the numerator and denominator are 0 (0/0) simultaneously, the numerator is zero and denominator is a number other than zero, the numerator is 1 and denominator is 0 (1/0), and the numerator is any number and denominator is 0 (for example, a/0). The answers of the PTs are given by being coded as 0/0=indefinite, 0/number=zero, 1/0=undefined, indefinite, infinite, no result, undefined-indefinite (Table 2). 5 participants explained that the expression 0/0(zero over zero)is indefinite by using the AMA and rule strategies. Some of the preservice teachers concentrated on the meaning of zero in the process of answering this question and expressed it as “nullity and nothing”. Moreover, while some of the preservice teachers who stated that 1/0 is infinite expressed this with the limit approach and indicated that it can be plus infinite or minus infinite according to the approach from the right and left to the number 0, others explained that 1/0 can be infinite using directly the anthology and rule strategies without explaining it through the limit approach. Upon examining the data, it was observed that students who graduated from the department of mathematics and received the pedagogical formation education used intensively the rules and mathematics teaching students used intensively the analogies for expressing the meaning of the division by zero and 1/0.

The preservice teachers who would explain the division by zero to secondary school or high school students used the AMA, analogy, and rule among the explanatory strategies and furthermore, 5 PTs thought the explaining as lecturing and explained how to they would teach the lesson and which methods and techniques they could use in general. Apart from that, 11 PTs left this question unanswered. Among these strategies, the rule was used the most and the AMA was used the least. Only a few of the preservice teachers who tried to explain the division by zero concentrated on the meaning of zero and stated that zero expresses the “nullity” and interpreted the division of a number by zero as undefined. Some of the preservice teachers stated that they would explain 1/0 to secondary school student as undefined and to high school students as infinite. They stated that this can be explained by the limit approach since there is a subject about the limit in high school. It was observed that mathematics teaching students would explain the divisibility by zero by using mostly the analogies and mathematics department students by using mostly the rules.

The Use of the Abstract Mathematical Argument (AMA)

This strategy is among strategies in which abstract mathematical thinkings are expressed with reasoned explanations. Using the AMA strategy, the PTs interpreted the meaning of expressions 0/0 and 1/0 as indefinite, undefined, and infinite. However, only one of the PTs explained to the students that 1/0 is undefined by using the AMA. It was observed that PTs using this strategy are mostly students of the department of mathematics receiving the pedagogical formation education.

Table 3 The AMA strategy explaining the divisibility by "0"

Categories	Sub-categories	Examples of Student Answers
The meaning of 0/0	Indefinite	*…0/0 is indefinite. However, in the case where 0=0.x, x can be given an infinite number of values, but since the number zero is an absorbing element, the value given will be indefinite... (T16, T26)
The meaning of 1/0	Indefinite	*0 cannot be divided by any number but itself and the result of 1/0 is indefinite. If 1/0=a, there is not a number a providing 1=0.a. (T12)
	Undefined	*1/0 is undefined. For example, it equals to number a. Since a/0=x $\Rightarrow$ a=x.0, a≠0, there is no number to give for x; therefore, it is undefined.(T7, T8) *1/0 is undefined because there is no such number providing 1=0.x…(T16, T26) *1/0 is undefined because if a≠0, a/0=x, and it will be a=0.x. Let's try to denominate x here. Since there is no such number that "gives a value other than zero when multiplied by zero", it is undefined.(T15) *Since x.0=0 if 1/0=x, x can get any value. Since $x \in (-\infty, +\infty)$, 1/0 is undefined. (T18) *The division of a number by 0 is considered undefined.For a/0, limits of 0 from the right and left are examined.Results can be obtained in the limit values of $\lim_{x\to 0+}\frac{a}{0}$ and $\lim_{x\to 0^-}\frac{a}{0}$. (by converging to 0) (T5)
	Infinite	*…On the subject of limit, we learned that the division of a number by 0 is infinite. There are infinite numbers between two numbers. We have to divide this one unit place by such a small number to obtain infinite numbers. Therefore, we divide it by zero and get infinite. It is observed that the notation changes from right to left in operations $1/0^+=+\infty$, $1/0^-=-\infty$. (T27, T37)
In the explanation process	Undefined (1/0)	*Since x.0=0 if 1/0=x, x can get any value. Therefore, 1/0=undefined. (T18)

Preservice teachers explained the division by *0* separately as the division of zero by zero and the division of 1 by zero. T16 and T26, who explained that 0/0 is indefinite, performed cross-multiplication based on the equality of this to any x number and obtained the result of 0=0.x. One of them stated that as zero is an absorbing element, x can be given infinite values, and because the value to be given is indefinite, x is indefinite, and the other one stated that x has infinite values and therefore, it is also indefinite. The PTs, who equated the expression 0/0 to x, stated that x has infinite numbers of values to get and therefore, since it is impossible to determine which value it will have, x is indefinite. Here, the PTs used the infinite and indefinite concepts interchangeably. 7 of the participants in this category interpreted 1/0 as undefined and 2 as infinite. T15 made an explanation as *"When a≠0, a/0 is undefined, … 1/0 is undefined because let's say when a≠0, it is a/0=x. Therefore, it will be a=0.x. Let's try to denominate x here. Since there is no such number that "gives a value other than zero when multiplied by zero", it is undefined".* T15 stated that in the case where the number a is different from zero, *a/0* is undefined and indicated that the number *1/0* is undefined. In order to reveal the undefinedness, he/she generalized number 1 to a different *a* number and tried to interpret the meaning of the number *a/0* and he/she used the AMA strategy for this. He/she equated the division of any *a* number by *0* to *x* and used the definition of proportion and performed cross-multiplication. As a result, he/she obtained an equation of *a=0.x*. Since there is no number that will give a value other than zero when multiplied by *0* in this result, he/she interpreted the divisibility by *0* as undefined. The majority of the participants using this strategy went from interpreting the divisibility by 0 to interpreting the meaning of *1/0* and stated that the result is undefined. In order to indicate why it is undefined, for example, T16 and T26 equated *1/0* any *x* number, and when they performed cross-multiplication, they came to a conclusion as *1=0.x.* Since there is no value that gives *1* when multiplied by *0* (this explanation was not made, but this result was reached from the explanation made), they stated that *1/0* is undefined. Furthermore, they explained that in the case where the numerator is *0* instead of *1(0/0)*, the result is indefinite. It was observed that some of the PTs who preferred this strategy made an error in the operations they performed and that they made reasoning based on the incorrect operation. For example, T18 equated 1/0 to x and as a result of the cross-multiplication wrote the equality as x.0=0. Here, he/she stated that x can have every value, thus he/she interpreted 1/0 as undefined. However, in this operation, the equation should have been written as x.0=1 and the interpretation should have been written according to this operation. However, even though the operation was incorrect since it was tried to prove it based on tangible mathematical reasoning and the mathematical definitions made included operations in the definition of divisibility by Arıkan and Halıcıoğlu (2012), the answers in this category were evaluated as the AMA. Some PTs, (T7 and T8), who think that 1/0 is undefined, also expressed this as “1/0=undefined”.

There were also two PTs who interpreted 1/0 as infinite. T27 and T37 stated that by the limit approach, the value of 1/0 is $+\infty$ or $-\infty$, respectively, according to the approach to 0 from the right and left. These PTs actually addressed the case of x approaching 0 from the right and left in the 1/x function and stated with the operations they performed that x cannot be exactly 0 but can have a value that is very close to zero, and in this case, 1/x can approach infinity. From this aspect, since the operations performed were based on abstract mathematical reasoning, they were evaluated in the AMA category.

The Use of Analogy

This strategy refers to the use of tangible contexts or intangible objects, that is, physical representations, in expressing reasoning when explaining any concept. It was observed that this strategy was mainly used by students studying at the department of mathematics teaching and only 2 PTs are students graduated from the department of the mathematics. Some PTs using the analogy interpreted the division by zero as undefined, infinite, and no result and others did not specify what 1/0 equals to and explained what the division by zero is and what it represents. In this process, the PTs who used the physical representation or tangible context sometimes divided the numerator by the denominator expressing “nullity, nothing” or by gradually increasing or decreasing numbers, and other times searched for zero within a whole or interpreted the number obtained by dividing the numerator by the denominator and addressed the negativity state of this number. It was observed that the tangible representations of the PTs who used this strategy were mostly the “cake” model. Apart from the cake, they used representations such as “sweets, kid, person, board, apple, knife, knife stroke, whole, object” or tangible contexts such as searching for something non-existent within any number (Table 4).

Table 4 The analogy strategy explaining the divisibility by "0"

Categories	Sub-categories	Dimensions	Examples of Student Answers
Undefined	Dividing the numerator by the denominator(nullity, nothing) (T34, T36, T46, T44)	Dividing a concrete object into nothing	…I would group students in a group of 5. I would give some sweets to each student. I would say that divide the sweets among yourselves with this much for each and divide the last 5 pieces of sweets into nobody. Since a whole cannot be divided by nullity, it is undefined...(T44)
		Negativity of the division	…The number of knife strokes required to divide a loaf of bread into 0 is 0-1=-1…As a result, the student will understand that the number cannot be divided by 0. (T34)
	Dividing the numerator by the denominator (T1, T37)	Dividing existing object into non-existent object	…Let's divide nothing we have into 5 people, that is (0/5). As a result, we can say that zero falls to everyone's share. On the other hand, if "1" is divided by "0" as 1/0, it is impossible to obtain any result as there is nothing to share the number "1" into. For example, 5 pieces of sweets cannot be divided into non-existent kids, that is, the nullity. (T1, T37)
	Dividing the numerator by gradually decreasing numbers (T30, T31, T33, T37, T42)		Students are shown how to divide a cake into 5 pieces. Then, it is shown how to divide a cake into 2 pieces. For the next step, by dividing it into 0 pieces, that is, by dividing into a little piece, that is, by decreasing the amount of flour in the cake, the infinity is achieved. (T30, T33, T37, T42)
Infinite	Dividing the numerator by gradually increasing numbers (T32)		…For example, if we first divide an apple into 2 with a knife, then divide each piece into 2 and those pieces into 2, and keep on dividing, we will obtain so many pieces. If we keep dividing the apple continuously in this way, we will get infinite numbers of pieces. ...In other words, 1/0 is infinite. (T32)
	Searching for zero within a whole (T35)		1/0 is the division of a whole by 0. In other words, it is searching for the number of zeros within a whole. …For example, in the operation 10/2, we see how many 2s there are in 10 and we find 2s in 6 and we say that the result is 3. Similarly, if we search for 0 in 1, we can say that the result is infinite. (T35)
No result	Dividing the numerator by the denominator(nullity, nothing) (T15, T44)	Dividing a concrete object into nullity (nothing)	Nothing is divided by zero. …For example, we cannot divide a cake into non-existent people or 5 pieces of sweets into non-existent people…(T44)
Other	Dividing the numerator by gradually decreasing numbers	Dividing a whole by zero	…For example, I would ask students to first divide a whole consisting of 6 equal pieces into 6, then into 3, 2, 1, respectively, show the pieces they have obtained as a result of the division. Finally, I would ask them to divide a whole into 0. When students cannot divide it, I would tell them that numbers cannot be divided by 0. (T28)
	Dividing the numerator by the denominator(nullity, nothing) (T31, T36)		… I would take an apple to the class and divide this apple into 3 pieces. There would be 3 pieces in my hand. I would give these pieces to one of my students. …I would say that since I do not have any apples and there is no such concept as a nullity in the mathematics, we use the number zero instead. …(T31, T36)

Dividing the numerator by gradually increasing numbers (T32, T45)	…We can explain it by making students realize that the number of pieces increases when we try to obtain as many small pieces as possible by dividing an apple or any object into pieces. We can show them that the more pieces we divide the apple into, the smaller pieces will be obtained. We will try to make them realize that as the pieces get smaller, their size will approach zero. (T32, T45)

Some of the preservice teachers thought that dividing a whole (a cake, an apple etc.) by zero is dividing it by nullity and nothing, but they interpreted the result in different ways. The preservice teachers interpreted the result as infinite (T31 and T37) made a similar explanation to each other, such as *"... the division of a number by 0 is ∞. Let's divide something we have not into 5 people, that is (0/5). As a result, we can say that everyone gets zero. On the other hand, if the number "1" is divided by "0" as 1/0, it is impossible to reach any conclusion as there is nobody to share the number "1". For example, we cannot divide five pieces of sweets into non-existent kids, that is, nothingness."* Here, the participants used tangible objects such as "sweets" and "kids" to interpret the division by 0. Although they expressed that the division of a number by *0* is infinite,they interpret the expression of *0/5* at the first step and indicated that *0/5* equals *0*. It was observed that the participants interpreted the division by 0 as *0/5*, that is, dividing *0* by a number. That is, the participants misinterpreted the *number/0* as *0/number*. In interpreting the number 1/0, they stated that there was nobody to share the number *1* and considered the denominator where the number *1* was present as *5* and interpreted the number *5/0*. They tangibly represented *5* present in the numerator as "sweets" and *0* present in the denominator as "non-existent kid" and "nothingness", and interpreted the result as the number of sweets for each kid. Here, it was stated that an object (sweets) cannot be divided by or shared with a non-existent thing, that is, nothingness (0) and this situation of not being able to divide was interpreted as infinite. Moreover, some of the preservice teachers who also thought that the result is infinite (T30, T31, T33), made explanations that would indicate that the denominator dividing the numerator gradually decreased. The PTs used the expressions *"When a number is divided by 0, the result will be infinite. ...Let's take a cake as an example. We can obtain infinite numbers of slices if we slice the cake thin enough. In the operation 1/0, consider 1 as a cake. As we divide 1 by as small numbers as we can, we will approach 0 further. The slices we obtain will also increase each time. Based on this logic, 1/0 is infinite…"*The PTs did not think the denominator here directly as zero and stated that in the case where the denominator gradually decreases starting from a large number, we will approach zero and the result to be obtained by the approach of the denominator to zero is gradually increasing and it is infinite. In fact, it can be said that the PT made a hidden limit operation. Although the PTs do not use the limit operations, the reasoning that he/she made can only be explained by the limit approach. However, it is not correct to say that the words that the PTs used and the situation they tried to explain exactly coincide with each other. This is because the PTs mentioned the process of dividing a cake in a way to get small slices. Here, in the expression "*...We can obtain infinite numbers of slices if we slice the cake thin enough."*, he/she talks about the number of cake slices being more than possible, and even the number of slices being infinite. Therefore, he/she states the need for the denominator to gradually decrease with the statement "*As we divide 1 by as small numbers as we can, we will approach 0 further*". However, in order to increase the number of slices, the number in the denominator should not be reduced but be increased gradually. However, the PAs made an explanation by ignoring this situation. The state of concretization here is formed by the incorrect conclusions of the PTs. But as an exact opposite of the case, T32 made an explanation as "*...Let's take a number as an example. This number is 20. Divide 20 by 20. 20/20=1. If we continue by decreasing the number of parts that are divided: 20/20=1, 20/10=2, 20/5=4, ..., 20/1=20. As can be seen, if we reduce the number that we divide by a number, the result we obtain increases. In that case, if we divide the number 20 by values approaching 0, the result always increases. When we divide it by 0, it will approach an infinite value"*. T32 used the limit approach within the correct reasoning by expressing that the result will always increase and approach infinite if the number in the denominator decreases and approaches zero. Likewise, there are also PTs who interpreted the division of the numerator by nullity, a non-existent thing as undefined and no result apart from infinite. For example, T36 and T46 interpreted that the expression is undefined by saying "*We cannot divide something by nothing. For* *example, we can divide a cake into 2 to 3 pieces, but we cannot divide it into 0 ...*", and T44 interpreted the division by zero as no result by saying *"Nothing is divided by zero... For example, we cannot divide a cake or 5 pieces of sweets into non-existent people. ...".*

A PT (T34) who interpreted 1/0 as indefinite made an expression as *"Dividing a whole into 2 equal pieces tangibly is dividing 1 by 2. But it is impossible... to divide a whole by 0. In order to divide a whole into 2, a knife stroke of minus 1 of the number of the pieces desired is required; however, since -1 knife stroke is required for 0 pieces, it is undefined...".* Here, the participant used tangible contexts and representations, including "whole", "piece", "knife", "knife stroke" for interpreting the number *1/0*.In order to interpret the number *1/0*, he/she first interpreted the number *1/2* and shifted from there to the case where the denominator is zero. It was stated that the number of knife strokes required to divide *1* whole into *2* equal pieces is as much as minus 1 of the desired number of pieces. He/she expressed that the number of piecesshould be *0* and the number of strokes should be *-1* in *1/0* and since the number of strokes cannot be negative, he/she interpreted *1/0* as undefined. As can be seen in these examples, analogies are actually limited to positive numbers. It is impossible to interpret, for example, number *-5/0* with these analogies.

Some PTs considered the number 0 in the denominator in 1/0 as nothing and interpreted it as the division of the number in the numerator by nothing, but did not state what the resulting expression would be like.

The Use of Rules

Rules are the state of definitions or theorems with mathematical significance. Rules are also used as explanatory strategies. However, this strategy refers to the mathematical realities that individuals remember and accept. That is, they explain the situations that individuals believe to exist. It was observed that the PTs use rule strategy the most on the division by zero. Furthermore, it was determined that the mathematics department graduates use this strategy more than the students studying at the department of mathematics teaching. Using these strategies, the PTs interpreted the division by zero as infinite, indefinite, undefined, undefined-indefinite, and zero to be able to divide by zero (Table 5).

Table 5 The rule strategy explaining the divisibility by "0"

Categories	Categories	Sub-categories	Examples of Student Answers
The meaning of division by zero	0/0	Indefinite	*…If a=0, a/0 is indefinite. (T6, T38)
	0/number	Zero	*0/number=0. (T24)
	1/0	Undefined	*1/0 is undefined… (T2, T9, T10, T11, T13, T14, T19, T38, T39, T40, T41, T48) *It is undefined because if we divide a number by nothing, we cannot reach a mathematical expression… (T23) (T29)
		Indefinite	*This number is a judgement that equals to infinity, as we have examined in integral. Such as the indefinity of 0.0.0.0…0=0.∞ , 1/0 is also indefinite…(T25)
		Infinite	*All numbers are divided by zero. If we consider in the limit situation of 1/0, it is ∞ . (T17, T20, T27) *1/0 = ∞ . (T21, T22, T28, T42, T45) *We learned in the primary school and high school that the division of any number by 0 is undefined….(T4) *Undefined = The result is - ∞ or + ∞ according to the negativity or positivity of the numerator. (T47)
		Undefined-Indefinite	*The number 0 can be divided by all real numbers but all real numbers are not divided by 0. The value will be undefined and indefinite. (T3)
		Other	*I think; all numbers must be divided by 0 because zero is above all numbers. We can calculate the value of 1/0 as the limit. (T43)

	a/0	Undefined	*Number/0 is undefined. (T9, T13, T14, T19, T21, T24, T38, T40)
		Indefinite	*…In secondary schools and high schools, 'nothing' is used instead of 0. Can we say that the division of a number by 0 means that the number is not divided at all?... (T6) *…5/0= ∞ …(T10)
		Infinite	*No numbers can be divided by 0. If they did, the result would be infinite because there are infinite numbers of0 in the number...(T22, T28)
		Zero	*n/0=0. If any n number is divided by 0, the result will be zero….(T35)
The explanation process	1/0	Undefined	*1/0 is undefined… (T38, T41)
		Infinite	*For high school students, 1/0 = ∞ . (T27)
		Undefined(Indefinite)	*I would say that there is no number that can be divided by 0 and the result will be indefinite (undefined). (Ö3)
	a/0	Undefined	*Number/0 is undefined. (T2, T7, T9, T10, T13, T16, T19, T20, T24, T38, T47) *I would explain to secondary school students that the result is undefined. I would say that consider this as we cannot divide a number by a non-existent number...(T27, T42)
		Indefinite	*…In the operation 17/0, how many zeros in 17? …An indefinity occurs in our minds, does not it?...(T25)
		Infinite	*I would say that there are infinite numbers of 0 in every number. (T22, T35)

In the answers in the rules category, there are answers in which the meanings of 0/0, 0/number, 1/0, and a/0 were interpreted separately. While 0/0 was interpreted as indefinite, 1/0 was interpreted as infinite, indefinite, undefined, undefined-indefinite, and zero. The interpretations in this category reflected what participants recalled based on previous learning. Upon examining the answers in this strategy, it was observed that the majority of the PTs expressed 1/0 as undefined with a memorized sentence. It was determined that the answers here are generally as "*1/0 is undefined. Such an expression cannot be divided by 0.", "... there is nothing in zero", "If we consider in the case of 1/0 limit, it is ∞ .".* It was found that these explanations were not based on any abstract mathematical technique, but rather were the knowledge that the participant believed and accepted. The answers in the rules category reflected the lack of participants' understanding of the question. A preservice teacher (T3) used the word undefined simultaneously in the same sense as the word indefinite and the others used the concepts of undefined and indefinite in different meanings.

In this strategy, one of the PTs (T47) stated that 1/0 is infinite, but made an explanation on the paper as *"undefined = The result is -∞ or +∞ according to the negativity or positivity of the numerator."* Upon taking into account this explanation, it was observed that the PT used the concept of undefined instead of the concept of infinite.

CONCLUSION AND DISCUSSION

In this study, the divisibility by zero, the meaning of 1/0, and how these concepts will be explained to secondary school/high school students. It was observed that the PTs used the rule the most and the AMA strategy the least. It was found that in explaining the divisibility by zero, some PTs addressed 0/0, 0/number, 1/0, and number/0 expressions separately and they usually interpreted these expressions as undefined, indefinite, infinite, zero, and no result.

Upon examining the answers given by the PTs, it was seen that they mostly used the rule strategy in the explanation in the meaning of 1/0 and the process of explaining it to their students. It was found that the PTs produced this expression themselves and tried to explain it as much as they recalled from what they previously learned, without any mathematical basis. This is a clear indication that the PTs cannot fully conceptually understand the divisibility by zero. Since the PTs do not have a sufficient mathematical understanding in this regard, they even explained it to their students by the rule and tried to make them memorize the rules. As a matter of fact, in some studies in the literature (Arsham, 2008; Ball, 1990; Cankoy, 2010; Crespo and Cynthia, 2006; Eisenhart et al., 1993; Quinn, Lamberg, and Perrin, 2008), the meaning of a ÷ 0 has also been questioned and the finding that the students and teachers lacked conceptual knowledge in this regard supported the finding in this study.

Another finding obtained in the research is the fact that the strategy mostly preferred by the PTs following the rule is the anthology. Moreover, it was determined that mathematics students studying at the department of mathematics use this strategy more than the students receiving the pedagogical formation education. This may be due to the fact that secondary school preservice teachers respond to the needs, interests, and needs of their students. That is because due to their age, secondary school students need more tangible experiences in terms of mental development than high school students. This may have led the secondary school mathematics preservice teachers to use their strategy preference in favor of the analogy where the concepts are described by concretizing. As a matter of fact, this finding also supports the finding by Karakuş (2017) that as the grade level increases, preservice teachers shift from tangible educational explanations to intangible educational explanations in explaining that the division of a number by zero is indefinite. Furthermore, Kinach (2002) stated that courses that PTs took in the university affect their explanatory strategy preferences. Therefore, this may also be due to the courses taken by the PTs.

In the study, it was observed that while the PTs explaining the concept by analogy expressed the expressions by concretizing, they made incorrect explanations in some cases and used real-life situations whose mathematical basis might be wrong. For example, in the expression of some PTs "*...we can obtain infinite numbers of slices if we slice the cake thin enough...As we divide 1 by as small numbers as we can, we will approach 0 further*", it was expressed that the number in the denominator is reduced to approach zero in order to obtain many slices of cake. However, the situation is exactly the opposite. In order to increase the number of slices, the number in the denominator needs to gradually increase. Here, the PT confused the size of a slice with the number of slices and was mistaken. A PT who makes such an explanation will confuse students and students will have difficulty in understanding the situation. Upon handling from this aspect, it seems very difficult to implement a teaching in which teachers can teach mathematics in accordance with the level of the students as long as the teachers do not have the proper mathematical knowledge, which they have learned with the reasons and equipment.

It was observed from the findings of the research that some of the preservice teachers (Table 5-T10) interpreted the meaning of 1/0 as undefined in the written form and interpreted as infinite symbolically or stated that it is undefined and later expressed that it is infinite (Table 5-T20, T21, T47). In other words, it was revealed that the PTs could not distinguish these two concepts from each other by the explanations that they made and that they thought that the concepts of indefinity and infinity had the same meaning. Some of the PTs stated that the concepts of undefinedness and indefinity had the same meaning by the expressions *"undefined(indefinite)"* they wrote on the paper. Furthermore, T16 and T26 stated that in cases of interpretation where 0/0 is undefined, 0/0=x and x can have an infinite number of values. However, they stated that the value of 0/0 that they expressed as x is indefinite. While these PTs gave an infinite number of values to the variable x, they interpreted this expression as indefinite because it is not possible to determine which of the infinite values is x. Here, upon examining the meanings that the PTs attributed to x, it is thought that they used the concepts of infinite and indefinite in the same meaning. Upon evaluating from this aspect, it can be said that the PTs do not clearly know the distinction between the indefinite and infinite, undefined and indefinite, and infinite and indefinite concepts, and could not distinguish them from each other. In a study they conducted, Jaffar and Dindyal (2011) showed that participants confused the concepts of undefinedness and indefinity and substituted these concepts for each other. Furthermore, in a research that they carried out with preservice teachers, Even and Tirosh (1995) found that teachers were able to distinguish the situations of indefinity and undefinedness from each other. However, the infinity is used as a place that cannot be reached or something that is too big (Nesin, 2002), undefinedness is used in situations where a proper result cannot be obtained while operating with a standard definition, and indefinity is used in situations where it cannot be determined which one of the different possible results are valid or in situations where different results are obtained by different methods (Ozmantar, 2008). Therefore, these three concepts are used in completely different meanings.

Although the majority of participants in the study correctly interpreted 1/0 as indefinite, there were also those who stated that the meaning of this expression is infinite and indefinite. This may have been due to the fact that the preservice teachers confused the undefined, indefinite, and infinite concepts. In fact, they may have actually made an explanation assuming that these three concepts are the same. Such that, T23 stated that 1/0 is undefined and made an explanation as "*The meaning of 1/0 is 1/0= 2, 3, 5, … There is no definition in the division of 1 by zero since all numbers can be written in the numerator…"*.Upon examining this explanation of T23, it was observed that he/she actually defined indefinity. However, T23 mentioned indefinity when expressing undefinedness. As a matter of fact, in research, Kanpolat (2010) obtained similar results with this finding.

In conclusion, it was observed that the knowledge of the PTs on the divisibility by zero was inadequate and the explanations were mostly in the form of rules. As a matter of fact, a similar situation was observed in some of the studies in the literature (Crespo and Nicol, 2006; Even and Tirosh, 1995). Only a few of the PTs explained the divisibility by zero using the AMA strategy. This showed that the PTs are very weak in terms of using reasoning techniques that provide abstract mathematical thinking and definition. Therefore, in the field and field education courses at university, the definitions of these concepts should be explained with reasons, and especially the meaning of each concept should be carefully emphasized. The teaching of abstract algebra to preservice teachers should be reconsidered and the PTs should be helped to re-learn numbers and operations significantly in order to prepare them to interpret abstract mathematical concepts in such a way support teaching practices.

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